

GEOMETRY OF T-DUALITY

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Abstract

A "reduced" differential geometry adapted to the presence of abelian isometries is constructed. Classical T-duality diagonalizes in this setting, allowing us to get conveniently the transformation of the relevant geometrical objects such as connections, pullbacks and generalized curvatures. Moreover we can induce privileged maps from the viewpoint of covariant derivatives in the target-space and in the world-sheet generalizing previous results, at the same time that we can correct connections and curvatures covariantly in order to have a proper transformation under T-duality.

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1 Introduction

T-duality is a fundamental tool in the understanding of the fashionable string dualities [1]. The elementary statement of the T-duality establishes that the perturbative spectrum of a string theory with a dimension compactified on a circle of radius R , is equivalent to the one compactified in a circle of radius $1/R$, provided we interchange winding and momentum quantum numbers at the same time that we transform the string coupling constant [8, 10]. If we allow the presence of a generic geometry (metric $G_{\mu\nu}$, torsion potential $B_{\mu\nu}$ and dilaton Φ) having an abelian Killing vector in the compactified direction X^0 , the backgrounds resulting to be 1-loop (conformally)equivalent are given by the Buscher's formulas [2]:

$$\begin{aligned}\tilde{G}_{00} &= 1/G_{00} \\ \tilde{G}_{0i} &= B_{0i}/G_{00} \quad \tilde{B}_{0i} = G_{0i}/G_{00} \\ \tilde{G}_{ij} &= G_{ij} - (G_{0i}G_{0j} - B_{0i}B_{0j})/G_{00} \\ \tilde{B}_{ij} &= B_{ij} - (G_{0i}B_{0j} - B_{0i}G_{0j})/G_{00} \\ \tilde{\Phi} &= \Phi - \frac{1}{2} \ln G_{00}\end{aligned}\tag{1}$$

In recent years a non-perturbative usage of T-duality has been made in the context of open strings playing with the map of the Neumann-Dirichlet boundary conditions. The promotion of the hypersurfaces in which strings rest their endpoints to be dynamical extended objects called Dirichlet-branes, allow their identification with the carriers of the RR-charges required by the string duality at the same time that it makes doubtful the name of "string theory" for the resulting scenario [4, 5].

Many features of this topic of T-duality have been developed extensively in the literature ([3, 6, 7, 8]). Despite that important effort, it seems to be a lack of a systematic study of the mapping between geometries for the stringy (1-loop) equivalent space-times. This gap is related with the nature of the non-linear map (1) which highly complicates the calculations for the transformation of geometrical objects such as the generalized connection and its curvature, privileged maps for the covariant derivatives and pullbacks, and many others.

In order to clarify all these points it is presented a sort of "parallel differential geometry" that we will call "Reduced Geometry", having the property of being adapted to the presence of abelian Killing vectors (in fact it is only defined in that context). We will see how T-duality transformations diagonalize in this setting for the main geometrical objects, including the generalized curvature ; for the later, we found for the first time its complete transformation, which can be expressed in a covariant way in terms of itself and of the Killing vector.

Moreover, a "canonical" T-duality transformation is constructed for arbitrary tensors with the property of transforming linearly the covariant derivatives calculated from the generalized connections. These results unify and generalize the map obtained for the p-forms in [6], and it includes the fundamental one of the complex structures for holomorphic Killing vectors [7].

We can extend the result giving above to the "canonical" T-duality transformation of maps from the world-sheet to the tangent space of our target-space manifold. The clasical string dynamics will be the most representative example of this "canonical" map.

In section 2 we define and describe the construction of the "reduced geometry" giving the basic map relating "usual" and "reduced" objects (generic tensors, connections and curvatures). In section 3 we get the "canonical" T-duality map, relating linearly covariant derivatives, and a "non-canonical" one relating linearly "covariant divergences". In section 4 we obtain the "canonical" map for the classical world-sheet dynamics. Section 5 shows the generalized curvatures' transformation and the minimal correction for them to transform linearly under T-duality. As a straightforward outcome I rederive the 1-loop beta function's transformation. In section 6 we found a "canonical" covariant derivative commuting with the "canonical" T-duality transformation. It is used to get a set of new T-duality scalars. In the Appendix we summarize the basic formulas.

2 Reduced Differential Geometry

In this section will be built a parallel tensor calculus for manifolds with an structure endowed with abelian Killing vectors. The main objective is to exploit the presence of these Killings in order to get a strongly simplified structure that I will call the Reduced Geometry. As we will see, that structure is nicely adapted to T-duality.

Let us assume the existence of a set of n commuting vector fields $\{K_a^\mu\}$ with $\{\mu, \nu = 0, 1, \dots, D-1\}$, $\{a, b = 1, \dots, n\}$ and D the dimension of the manifold M .

We restrict our attention to the space Ω of tensors V in M satisfying

$$\mathcal{L}_{k_a} V = 0 \quad (2)$$

that means simply that we can choose coordinates $\{x^i, x^a\}$ with $i = n, \dots, D-1$, called adapted coordinates, in which V does not depend on x^a , ie., $\partial_a V = 0$.

The covariant differentiation is not a mapping in Ω , or in other words, its commutator with the Lie derivative is in general non vanishing :

$$[\mathcal{L}_{k_a}, \nabla_\rho] V_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l} = - \sum_{s=1}^m (\mathcal{L}_{k_a} \nabla)_{\rho \nu_s}^\sigma V_{\nu_1, \dots, \sigma, \dots, \nu_m}^{\mu_1, \dots, \mu_l} + \sum_{r=1}^l (\mathcal{L}_{k_a} \nabla)_{\rho \sigma}^{\mu_r} V_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \sigma, \dots, \mu_l} \quad (3)$$

where I have defined ²

$$(\mathcal{L}_{k_a} \nabla)_{\mu \nu}^\sigma \equiv K_a^\alpha R_{\alpha \mu \nu}^\sigma + \nabla_\mu \nabla_\nu K_a^\sigma + 2 \nabla_\mu (K_a^\alpha T_{\alpha \nu}^\sigma) \quad (4)$$

being $R_{\alpha \mu \nu}^\sigma$ the curvature for the connection $\Gamma_{\lambda \beta}^\delta$ and $T_{\mu \nu}^\rho$ the corresponding torsion. It can be checked that (4) reduces to the desired $\partial_a \Gamma_{\mu \nu}^\sigma$ in adapted coordinates.

If we are interested in connections preserving the condition (2), we must impose

$$\mathcal{L}_{k_a} \nabla = 0 \quad (5)$$

Then, I have established our framework throught the conditions (2) and (5). Moreover I assume the choice of adapted coordinates to the Killings. There is a freedom for that choice that is reflected in the existence of a subset of diffeomorphisms (adapted diffeomorphisms)

²the conventions for the curvature can be found in the Appendix

relating the different possibilities. Modulo arbitrary changes in the x^i transverse coordinates, the relevant adapted ones are

$$\begin{aligned} x'^i &= x^i \\ x'^a &= x^a + \Lambda^a(x^j) \end{aligned} \quad (6)$$

Tensors in Ω transforms linearly under this change as is expected

$$V(x^i)_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l} = J(\partial\Lambda)_{\nu_1, \dots, \nu_m; \beta_1, \dots, \beta_l}^{\mu_1, \dots, \mu_l; \alpha_1, \dots, \alpha_m} V(x^j)_{\alpha_1, \dots, \alpha_m}^{\beta_1, \dots, \beta_l} \quad (7)$$

where I define

$$J(\partial\Lambda)_{\nu_1, \dots, \nu_m; \beta_1, \dots, \beta_l}^{\mu_1, \dots, \mu_l; \alpha_1, \dots, \alpha_m} \equiv \prod_{r=1}^l J_{\beta_r}^{\mu_r}(\partial\Lambda) \prod_{s=1}^m J_{\nu_s}^{\alpha_s}(-\partial\Lambda) \quad (8)$$

$$J_{\nu}^{\mu}(\partial\Lambda) = \delta_{\nu}^{\mu} + \delta_a^{\mu} \partial_i \Lambda^a \delta_{\nu}^i \quad (9)$$

Using a short notation

$$V' = J(\partial\Lambda)V \quad (10)$$

Because the abelian nature of the diffeomorphism, J provides a representation of $U(1)^n$ in the vector space of tensors of the same rank than V satisfying (2); in particular, $J(\partial\Lambda_1)J(\partial\Lambda_2) = J(\partial(\Lambda_1 + \Lambda_2))$ and $J(0) = \mathbf{1}$ implying $J^{-1}(\partial\Lambda) = J(-\partial\Lambda)$.

In the cases we are interested, it is natural to find a set of "transverse" gauge fields $\{A_i^a(x^j)\}$ transforming under the adapted diffeomorphism as

$$A_i'^a(x^j) = A_i^a(x^j) - \partial_i \Lambda^a(x^j) \quad (11)$$

Now I can define the "reduced tensor" v associated to V

$$v \equiv J(A)V \quad (12)$$

which has the property of being invariant under the adapted diffeomorphism (6).

$$v' = J(A - \partial\Lambda)J(\partial\Lambda)V = v \quad (13)$$

It makes sense to think about reduced tensors as the ones in a D-dimensional manifold with a dimension locally shrunk to a point. They keep their indices corresponding to the colapsed dimension, which are inert to the adapted diffeomorphisms, but they are sensitive to other transformations, as we will see in the T-duality case.

Looking at the explicit form of the matrix J, it is clear that the operation giving "reduced tensors" commutes with linear combinations, tensor products, contraction and permutation of indices.

Before following with this logic development let us see the most significant example. If we have a Riemmanian manifold with n commuting Killings

$$G_{\mu\nu} = \begin{pmatrix} G_{ab} & A_{ai} \\ A_{bj} & \hat{G}_{ij} + A_{ic}A_{jd}G^{cd} \end{pmatrix}$$

$$\mathcal{L}_{k_a} G_{\mu\nu} = 0 \quad (14)$$

where G^{ab} is the inverse matrix of G_{ab} . It is well known that the desired "transverse" gauge fields are

$$A_i^a(x^j) = G^{ab} A_{bi}(x^j) \quad (15)$$

Using the convention of writing the usual tensors in capital letters and the reduced ones in small letters, the reduced metric takes the simple form:

$$g_{\mu\nu} = \begin{pmatrix} G_{ab} & 0 \\ 0 & \hat{G}_{ij} \end{pmatrix}$$

If we keep ourselves with the conditions (2) and (5), we can repeat the arguments given above and conclude there exists the corresponding reduced covariant derivative ³:

$$\nabla v \equiv J(A) \nabla V \quad (16)$$

Taking account the explicit expression for J in every tensor representation, we can read off the reduced connection γ ⁴:

$$\gamma_{\mu\nu}^\rho = J(-A)_\mu^\alpha J(-A)_\nu^\beta J(A)_\delta^\rho (\Gamma_{\alpha\beta}^\delta - \partial_\alpha J(A)_\beta^\delta) \quad (17)$$

In an arbitrary choice of the "transverse" gauge field A_i^a the resulting reduced connection could not have any advantage, but with a "natural" choice, ie., (15) in a Riemmanian manifold, it is a very simplified version of the usual one. As the most significant example I will write the reduced connection for the Levi-Civita connection for the metric in (14)

$$\begin{aligned} \gamma_{ab}^c &= 0; & \gamma_{ab}^i &= -\frac{1}{2} \hat{\partial}^i G_{ab} \\ \gamma_{ia}^b &= \gamma_{ai}^b = \frac{1}{2} G^{bc} \partial_i G_{ca}; & \gamma_{ij}^k &= \hat{\Gamma}_{ij}^k \\ \gamma_{ja}^i &= \gamma_{aj}^i = \frac{1}{2} G_{ab} \hat{F}(A)_j^{bi}; & \gamma_{ij}^a &= -\frac{1}{2} F(A)_{ij}^a \end{aligned} \quad (18)$$

where hatted objects are the ones calculated with the quotient metric \hat{G}_{ij} ⁵, and $F(A)_{ij}^a \equiv \partial_i A_j^a - \partial_j A_i^a$ is the field strength of the gauge fields. The usual Levi-Civita connection for this case is written in the Appendix and the comparison shows the great advantage of using the reduced one. In the case of an arbitrary covariant derivation, we can get the corresponding reduced connection adding the reduced tensor of the additional one to the Levi-Civita connection (17).

Despite its simplicity, even in the simplest case, the reduced connection (18) has a very rich structure because the presence of torsion at the same time that a non-Levi-Civita symmetric part, both restricted by the necessary covariantly constancy of the reduced metric.

³the reduced covariant derivative is denoted by ∇ too. The distinction is made looking over what kind of tensor acts.

⁴details in the Appendix

⁵ $\hat{F}_i^{bj} = \hat{G}^{jk} F_{ik}^b$; $\hat{\Gamma}_{ij}^k = \frac{1}{2} \hat{G}^{kl} (\partial_i \hat{G}_{lj} + \partial_j \hat{G}_{il} - \partial_l \hat{G}_{ij})$

With the definition giving above of reduced covariant derivative the operation that gives the reduced tensors commutes with the basic operations of the tensor calculus: linear combination, tensor product, contraction, permutation of indices and covariant derivation. That feature together with its simplicity is the reason to call the whole setting the "reduced geometry".

The generic presence of torsion is the responsible for a little subtlety in calculating the reduced curvature. To see that, let us start with the Riemann-Christoffel curvature. Due to the commutation of reduced covariant differentiation with the reduced mapping, we should write:

$$[\nabla_\mu, \nabla_\nu]W_\rho = J_{\mu\nu\rho}^{\lambda\delta\pi}(A)[\nabla_\lambda, \nabla_\delta]w_\pi \quad (19)$$

for an arbitrary one-form W belonging to Ω , and its reduced version w .

Substituting commutators in both sides and taking account the presence of torsion in the second one, we get :

$$R(\Gamma_{L-C})_{\mu\nu\rho}^\sigma W_\sigma = J_{\mu\nu\rho}^{\lambda\delta\pi}(A)(R(\gamma_{l-c})_{\lambda\delta\pi}^\eta w_\eta + 2T(\gamma_{l-c})_{\lambda\delta}^\eta \nabla_\eta w_\pi) \quad (20)$$

I denote γ_{l-c} the reduced Levi-Civita connection (18) , $T(\gamma_{l-c})$ the associated torsion and Γ_{L-C} the Levi-Civita connection. At first glance it seems to exist an obstruction to identify the reduced curvature by the presence of the torsion term. The little paradox solves realizing the only non vanishing torsion is γ_{ij}^a (18) and therefore that contribution does not contain derivatives of w (because $\partial_a w = 0$) and can be added to the standart curvature. The resulting reduced curvature is :

$$(r_{l-c})_{\lambda\delta\pi}^\eta = R(\gamma_{l-c})_{\lambda\delta\pi}^\eta - 2T(\gamma_{l-c})_{\lambda\delta}^a (\gamma_{l-c})_{a\pi}^\eta \quad (21)$$

In the general case we can write the connection as $\Gamma = \Gamma_{L-C} + H$. Using the formula

$$\begin{aligned} R(\Gamma + Q)_{\mu\nu\sigma}^\rho &= R(\Gamma)_{\mu\nu\sigma}^\rho + \\ \nabla_\mu Q_{\nu\sigma}^\rho - \nabla_\nu Q_{\mu\sigma}^\rho - Q_{\mu\sigma}^\alpha Q_{\nu\alpha}^\rho + Q_{\nu\sigma}^\alpha Q_{\mu\alpha}^\rho + 2T(\Gamma)_{\mu\nu}^\alpha Q_{\alpha\sigma}^\rho \end{aligned} \quad (22)$$

being ∇_μ the covariant derivative calculated from the generic Γ connection and $T(\Gamma)$ the associated torsion, and using the properties of the reduced map, we get for the reduced curvature :

$$r_{\lambda\delta\pi}^\eta = R(\gamma_{l-c} + h)_{\lambda\delta\pi}^\eta - 2T(\gamma_{l-c})_{\lambda\delta}^a (\gamma_{l-c} + h)_{a\pi}^\eta \quad (23)$$

where $h_{\mu\nu}^\rho$ is the reduced tensor corresponding to $H_{\mu\nu}^\rho$.

In the present work we are interested in the basic $U(1)$ duality. There, the reduced metric

$$g_{\mu\nu} = \begin{pmatrix} G_{00} & 0 \\ 0 & \hat{G}_{ij} \end{pmatrix}$$

has the additional advantage of having almost a trivial transformation under T-duality (1);

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} \frac{1}{G_{00}} & 0 \\ 0 & \hat{G}_{ij} \end{pmatrix}$$

It is not relevant here to make an exhaustive catalogue of the reduced geometry. The reduced exterior differentiation and the reduced Lie-derivative mapping can be obtained writing them in terms of covariant derivatives (for example with the Levi-Civita connection); in such a way the reduced expression is manifest across (16) and coincides with the usual one except for the presence of torsion in the reduced Levi-Civita connection. The case of the differential of a two-form, which is relevant to the $U(1)$ T-duality calculations is done.

Given a two-form B , its exterior differential, can be written in terms of the covariant one calculated with the Levi-Civita connection as

$$H_{\mu\nu\rho} \equiv \frac{1}{2}(\nabla_\mu B_{\nu\rho} + \nabla_\nu B_{\rho\mu} + \nabla_\rho B_{\mu\nu}) \quad (24)$$

In those terms, the reduced H , i.e. h , is

$$h_{\mu\nu\rho} \equiv \frac{1}{2}(\nabla_\mu b_{\nu\rho} + \nabla_\nu b_{\rho\mu} + \nabla_\rho b_{\mu\nu}) = \frac{1}{2}(\partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu}) - T(\gamma_{l-c})_{\mu\nu}^0 b_{0\rho} - T(\gamma_{l-c})_{\nu\rho}^0 b_{0\mu} - T(\gamma_{l-c})_{\rho\mu}^0 b_{0\nu} \quad (25)$$

As we see the expression of the reduced version in terms of ordinary derivatives gives an additional term due to the presence of torsion in the reduced connection. Despite this apparent setback the reduced H has again nice properties from the T-duality point of view:

$$h_{0ij} = -\frac{1}{2}F(b)_{ij} \quad (26)$$

$$h_{ijk} = \hat{h}_{ijk} \quad (27)$$

where $F(b)_{ij} \equiv F(B_{0k} = b_{0k} = \tilde{G}_{0k}/\tilde{G}_{00} = \tilde{A}_k)_{ij}$ ⁶. I label the ijk component with a hat because it is T-duality selfdual:

$$\tilde{h}_{0ij} = -\frac{1}{2}F(g)_{ij} \quad (28)$$

$$\tilde{h}_{ijk} = \hat{h}_{ijk} \quad (29)$$

where $F(g)_{ij} = F(A_k = G_{0k}/G_{00})$.

The main purpose of the remaining sections is to exploit the power of the reduced framework in its application for the study of the classical geometry of the T-duality mapping.

3 Generalized T-Duality Mapping

The natural connections defined in the context of T-duality are Γ^\pm , with their reduced partners γ^\pm

⁶ $\hat{h}_{ijk} = \frac{1}{2}(\partial_i \hat{b}_{jk} + \partial_k \hat{b}_{ij} + \partial_j \hat{b}_{ki}) + \frac{1}{4}(F(g)_{ij} B_{0k} + F(b)_{ij} \frac{G_{0k}}{G_{00}} + F(g)_{ki} B_{0j} + F(b)_{ki} \frac{G_{0j}}{G_{00}} + F(g)_{jk} B_{0i} + F(b)_{jk} \frac{G_{0i}}{G_{00}})$ and $\hat{b}_{ij} \equiv B_{ij} - (G_{0i} B_{0j} - G_{0j} B_{0i})/2G_{00}$ is the T-duality invariant $\tilde{\hat{b}}_{ij} = \hat{b}_{ij}$ transverse torsion potential.

$$\Gamma_{\mu\nu}^{\pm\rho} = \Gamma(L - C)_{\mu\nu}^{\rho} \pm H_{\mu\nu}^{\rho} \quad (30)$$

$$\gamma_{\mu\nu}^{\pm\rho} = \gamma(l - c)_{\mu\nu}^{\rho} \pm h_{\mu\nu}^{\rho} \quad (31)$$

where the torsion $H_{\mu\nu\sigma} = H_{\mu\nu}^{\rho} G_{\rho\sigma}$ is as in (26). The explicit expressions for the reduced connection are gratefully simplified ⁷:

$$\begin{aligned} \gamma_{00}^{\pm 0} &= 0 \\ \gamma_{00}^{\pm i} &= -\frac{1}{2} \hat{\partial}^i G_{00} \\ \gamma_{i0}^{\pm 0} &= \gamma_{0i}^{\pm 0} = \frac{1}{2} \partial_i \ln G_{00} \\ \gamma_{ij}^{\pm k} &= \hat{\Gamma}_{ij}^k \pm \hat{h}_{ij}^k \\ \gamma_{0j}^{\pm i} &= \frac{1}{2} (G_{00} \hat{F}(g)_j^i \mp \hat{F}(b)_j^i) \\ \gamma_{j0}^{\pm i} &= \frac{1}{2} (G_{00} \hat{F}(g)_j^i \pm \hat{F}(b)_j^i) \\ \gamma_{ij}^{\pm 0} &= -\frac{1}{2} (F(g)_{ij} \pm G_{00}^{-1} F(b)_{ij}) \end{aligned} \quad (32)$$

Now it is easy to read off their T-duality transformation, that results to be diagonal:

$$\begin{aligned} \tilde{\gamma}_{00}^{\pm i} &= -(\frac{1}{G_{00}})^2 \gamma_{00}^{\pm i} \\ \tilde{\gamma}_{i0}^{\pm 0} &= \tilde{\gamma}_{0i}^{\pm 0} = -\gamma_{i0}^{\pm 0} \\ \tilde{\gamma}_{0j}^{\pm i} &= \mp \frac{1}{G_{00}} \gamma_{0j}^{\pm i} \\ \tilde{\gamma}_{j0}^{\pm i} &= \pm \frac{1}{G_{00}} \gamma_{j0}^{\pm i} \\ \tilde{\gamma}_{ij}^{\pm 0} &= \pm G_{00} \gamma_{ij}^{\pm 0} \\ \tilde{\gamma}_{ij}^{\pm k} &= \gamma_{ij}^{\pm k} \end{aligned} \quad (33)$$

Except for the $\gamma_{i0}^{\pm 0}$ component, we can arrange it in the way

$$\tilde{\gamma}_{\mu\nu}^{\pm\rho} = (-1)^{g_\mu} (\pm G_{00})^{(n^0 - n_0)} \gamma_{\mu\nu}^{\pm\rho} \quad (34)$$

where I define $g_0 \equiv 1$ if the μ index is contravariant, $g_0 \equiv -1$ if it is covariant and $g_i \equiv 0$ in both cases; Moreover $(n^0 - n_0)$ in front of an object (connection or tensor's component) with indices α , means $\sum_\alpha g_\alpha$ ⁸. The $\gamma_{i0}^{\pm 0}$ component transforms flipping the sign with respect to (34), which will be relevant in what follows.

It will be important too, to notice $\gamma_{i0}^{\pm 0}$ acts as a connection for (34)-type transformations : let the generic T-duality transformation in the reduced setting $\tilde{\Theta}^\pm = (\pm G_{00})^{\Delta_{\Theta^\pm}} \Theta^\pm$ for a given function $\Theta^\pm(X^j)$, there is a natural covariant derivative :

⁷ $\hat{h}_{ij}^k = \hat{h}_{ijl} \hat{G}^{lk}$

⁸I call it $n^0 - n_0$ because it reduces to the number of contravariant components being zero minus the number of zero covariant ones.

$$d_i \Theta^\pm \equiv (\partial_i + \Delta_{\Theta^\pm} \gamma_{i0}^{\pm 0}) \Theta^\pm \quad (35)$$

transforming as the Θ^\pm itself ⁹

$$\tilde{d}_i \tilde{\Theta}^\pm = (\pm G_{00})^{\Delta_{\Theta^\pm}} d_i \Theta^\pm \quad (36)$$

The simplicity of the reduced connection's transformation allow us to realize there is a diagonal T-duality transformation of reduced tensors that maps (again diagonally in the reduced setting) the usual (target-space) ∇_μ^\pm covariant derivatives. To see this, let us write them in terms of the (35) :

$$\begin{aligned} \nabla_0^\pm v_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l} &= \left\{ - \sum_{s=1}^m \gamma_{0\nu_s}^{\pm\sigma} v_{\nu_1, \dots, \sigma, \dots, \nu_m}^{\mu_1, \dots, \mu_l} + \sum_{r=1}^l \gamma_{0\sigma}^{\pm\mu_r} v_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \sigma, \dots, \mu_l} \right\} \\ \nabla_i^\pm v_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l} &= \left\{ d_i v_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l} - \sum_{s=1/\nu_s \neq 0}^m \gamma_{i\nu_s}^{\pm\sigma} v_{\nu_1, \dots, \sigma, \dots, \nu_m}^{\mu_1, \dots, \mu_l} + \sum_{r=1/\mu_r \neq 0}^l \gamma_{i\sigma}^{\pm\mu_r} v_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \sigma, \dots, \mu_l} - \right. \\ &\quad \left. \sum_{s=1/\nu_s=0}^m \gamma_{i0}^{\pm j} v_{\nu_1, \dots, j, \dots, \nu_m}^{\mu_1, \dots, \mu_l} + \sum_{r=1/\mu_r=0}^l \gamma_{ij}^{\pm 0} v_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, j, \dots, \mu_l} \right\} + \\ &\quad ((n^0 - n_0) - \Delta_{v_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l}}) \gamma_{i0}^{\pm 0} v_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l} \end{aligned} \quad (37)$$

The last term in (37) transforms with an undesired -1 with respect to the dominant $d_i v$, fixing the weight $\Delta_{v_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l}} = (n^0 - n_0)$. Therefore the reduced transformation is

$$\tilde{v}_{\nu_1, \dots, \nu_m}^{\pm \mu_1, \dots, \mu_l} = (\pm G_{00})^{(n^0 - n_0)} v_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l} \quad (38)$$

giving the linear map for the covariant derivatives under T-duality ¹⁰

$$\tilde{\nabla}_\rho^\pm \tilde{v}_{\nu_1, \dots, \nu_m}^{\pm \mu_1, \dots, \mu_l} = (-1)^{g_\rho} (\pm G_{00})^{(n^0 - n_0)'} \nabla_\rho^\pm v_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l} \quad (39)$$

Before writing the transformations in the usual (non-reduced) setting, I express them in a compressed notation as

$$\tilde{v}^\pm \equiv D_\pm(G_{00})v \quad (40)$$

$$\tilde{\nabla}^\pm \tilde{v}^\pm \equiv D_{\nabla}^\pm(G_{00})\nabla^\pm v \quad (41)$$

D_\pm and D_{∇}^\pm being diagonal matrices in every tensor representation.

Inverting (12) for v and \tilde{v}^\pm taking account $A_i = G_{0i}/G_{00}$ and $\tilde{A}_i = B_{0i}$ we get for \tilde{V}^\pm and $\tilde{\nabla}^\pm \tilde{V}^\pm$:

$$\tilde{V}^\pm = J(-B_{0i})D_\pm(G_{00})J(G_{0i}/G_{00})V \quad (42)$$

$$\tilde{\nabla}^\pm \tilde{V}^\pm = J(-B_{0i})D_{\nabla}^\pm(G_{00})J(G_{0i}/G_{00})\nabla^\pm V \quad (43)$$

⁹Moreover d_i satisfies the Leibniz's rule $d_i(A * B) = (d_i A) * B + A * (d_i B)$ and the "covariant constancy" of G_{00} , i.e., $d_i G_{00} = 0$.

¹⁰It must be stressed, to do not overcarry the notation, I always write $n^0 - n_0$, but in every case its value is given by the tensor's (connection's) component in front of it in the way described above.

Because J , D_\pm and $D_{\tilde{\nabla}}^\pm$ factorise, the T-duality mapping does too and can be written in terms of the matrices T^\pm and T_\pm defined below as:

$$\tilde{V}_{\nu_1, \dots, \nu_m}^{\pm \mu_1, \dots, \mu_l} = \left(\prod_{r=1}^l T_{\pm \beta_r}^{\mu_r} \right) \left(\prod_{s=1}^m T_{\nu_s}^{\pm \alpha_s} \right) V_{\alpha_1, \dots, \alpha_m}^{\beta_1, \dots, \beta_l} \quad (44)$$

$$\tilde{\nabla}_\rho^\pm \tilde{V}_{\nu_1, \dots, \nu_m}^{\pm \mu_1, \dots, \mu_l} = T_\rho^\mp{}^\lambda \left(\prod_{r=1}^l T_{\pm \beta_r}^{\mu_r} \right) \left(\prod_{s=1}^m T_{\nu_s}^{\pm \alpha_s} \right) \nabla_\lambda^\pm V_{\alpha_1, \dots, \alpha_m}^{\beta_1, \dots, \beta_l} \quad (45)$$

with

$$T_\mu^{\pm \nu} = \begin{pmatrix} \pm \frac{1}{G_{00}} & 0 \\ (\pm B_{0i} - G_{0i})/G_{00} & \delta_j^i \end{pmatrix}$$

$$T_{\pm \nu}^\mu = \begin{pmatrix} \pm G_{00} & \pm G_{0i} - B_{0i} \\ 0 & \delta_j^i \end{pmatrix}$$

where ν is the column index and μ is the row index. These matrices T_\pm and T^\pm , introduced by Hassan to the study of the T-duality of the extended supersymmetry [7], can be thought as a sort of "vielvein" relating indices of the initial and dual geometries. In what follows I will call (44) T-duality canonical transformation for the tensor V . Let us note (45) is not canonical. This anomaly is the responsible for the generalized curvatures of original and dual geometries do not transform simply changing indices with the T^\pm vielveins. We will see it in detail in the last section.

It must be stressed there are other tensors whose T-duality transformation is not as in (44); the most appealing cases are the torsion potential $B_{\mu\nu}$ and its field strength, the torsion $H_{\alpha\beta\gamma}$ [7]. It seems the natural place for the torsion is taking part of the generalized connection; then from (44)(45) or (33) we can extract the whole transformation to be :

$$\tilde{\Gamma}_{\mu\nu}^{\pm \rho} = T_\mu^\mp{}^\lambda T_\nu^\pm{}^\beta T_{\pm \alpha}^\rho \Gamma_{\lambda\beta}^{\pm \alpha} + (\partial_\mu T_\nu^{\pm \beta}) T_{\pm \beta}^\rho \quad (46)$$

For tensors covariantly constant with respect to one of both derivatives ∇_μ^\pm , the T-duality mapping is enforced to be given by (44). This is the case of the metric itself because $\nabla_\mu^\pm G = \tilde{\nabla}_\mu^\pm \tilde{G} = 0$. If we have in the manifold two covariantly constant p-forms A^\pm satisfying $\nabla_\mu^\pm A^\pm = 0$, there is a W-algebra in the underlying string sigma-model.¹¹ Another dual W-algebra is present in the dual string theory provided the p-forms transforms as in (44)[6]:

$$\tilde{A}_{0i_1 \dots i_{p-1}}^\pm = \pm \frac{1}{G_{00}} A_{0i_1 \dots i_{p-1}}^\pm$$

$$\tilde{A}_{i_1 \dots i_p}^\pm = A_{i_1 \dots i_p}^\pm + \sum_{s=1}^p \frac{(\pm b_{0i_s} - g_{0i_s})}{G_{00}} A_{i_1 \dots 0 \dots i_p} \quad (47)$$

If we have another Killing $K_\mu^{(2)}$ (non necessarily commuting with the one used to T-dualize) giving rise to a chiral current, it holds [9] $\nabla_\mu^\pm K_\nu^{(2)} = 0$ (+ for an holomorphic

¹¹That includes the mapping of complex structures , although it requires the additional vanishing of the Nijenhuis tensor [7]

current and – for an antiholomorphic one). The T-duality canonical transformation of $K_\mu^{(2)}$ gives another dual Killing vector with chiral current.

Instead of the privileged T-duality mapping for the generalized covariant derivative, we could be interested in the one for the "generalized divergence" $\nabla_\lambda^\pm Q_{\pm\nu_1,\dots,\nu_m}^{\lambda\mu_1,\dots,\mu_l}$. Again the study is very simplified in the reduced framework.

After determining the T-duality weight of $q_{\pm\nu_1,\dots,\nu_m}^{\lambda\mu_1,\dots,\mu_l}$ following the same procedure as in the covariant derivatives' map, the required mapping results to be :

$$\begin{aligned}\tilde{q}_{\pm\nu_1,\dots,\nu_m}^{\lambda\mu_1,\dots,\mu_l} &= (-1)^{g_\lambda} (\pm G_{00})^{(n^0-n_0+1)} q_{\nu_1,\dots,\nu_m}^{\lambda\mu_1,\dots,\mu_l} \\ \tilde{\nabla}_\lambda^\pm \tilde{q}_{\pm\nu_1,\dots,\nu_m}^{\rho\mu_1,\dots,\mu_l} &= (\pm G_{00})^{(n^0-n_0)'+1} \nabla_\lambda^\pm q_{\nu_1,\dots,\nu_m}^{\rho\mu_1,\dots,\mu_l}\end{aligned}\quad (48)$$

We have learned to read it in the common language as ($K^2 = K_\mu K^\mu$)

$$\begin{aligned}\tilde{Q}_{\pm\nu_1,\dots,\nu_m}^{\lambda\mu_1,\dots,\mu_l} &= K^2 T_{\mp\rho}^\lambda (\prod_{r=1}^l T_{\pm\beta_r}^{\mu_r}) (\prod_{s=1}^m T_{\nu_s}^{\pm\alpha_s}) Q_{\alpha_1,\dots,\alpha_m}^{\rho\beta_1,\dots,\beta_l} \\ \tilde{\nabla}_\lambda^\pm \tilde{Q}_{\pm\nu_1,\dots,\nu_m}^{\lambda\mu_1,\dots,\mu_l} &= K^2 (\prod_{r=1}^l T_{\pm\beta_r}^{\mu_r}) (\prod_{s=1}^m T_{\nu_s}^{\pm\alpha_s}) \nabla_\rho^\pm Q_{\alpha_1,\dots,\alpha_m}^{\rho\beta_1,\dots,\beta_l}\end{aligned}\quad (49)$$

This section shows how elementary geometrical objects transforms under T-duality depending on their defining properties : covariantly constant tensors transforms canonically (44), i.e., changing indices with the "vielbeins" T_\pm and T^\pm ; "divergenceless" tensors do it under (49) changing the index corresponding with the divergence with T_\mp instead of T_\pm ; the generalized connection transforms as a true connection (46) with respect to the T-duality canonical transformation with the only peculiarity of using T_\mp instead of T_\pm in the index corresponding to the derivation. This anomaly propagates to the T-duality transformation of every index associated to derivation as we have seen in (44)(49), and finally it will be the responsible for the inhomogeneous transformation of the generalized curvature, as we will show in the fifth section.

4 T-Duality Classical Dynamics

In addition to the D-dimensional manifold M representing the target space-time, in the context of strings we have a two dimensional world embedded in it, the world-sheet Σ , identified with the dynamical string ¹². At tree-level in the string dynamics Σ has the topology of the sphere. Choosing light-cone real coordinates σ^\pm , the covariant world-sheet derivatives for mappings $Y(\sigma^+, \sigma^-)$ between Σ and the tangent space of our manifold M in $X^\mu(\sigma^+, \sigma^-)$ are:

$$\nabla_\pm = \partial_\pm + \Gamma_\pm \quad (50)$$

being the pull-back in terms of the string's embedding $X^\mu(\sigma^+, \sigma^-)$

$$\Gamma_{\pm\mu}^\rho = \partial_\pm X^\alpha \Gamma_{\alpha\mu}^{\mp\rho} \quad (51)$$

Extending the definition (12) to the Y mappings

¹² I omit here p-branes, D-branes and any other kind of stringy extended objects.

$$y(\sigma^+, \sigma^-) \equiv J(A(X(\sigma^+, \sigma^-)))Y(\sigma^+, \sigma^-) \quad (52)$$

where y is the reduced mapping. With that definition we can proof the reduced partner of the pull-back is

$$\gamma_{\pm\mu}^\rho = (\partial_\pm x)^\beta \gamma_{\beta\mu}^{\mp\rho} \quad (53)$$

where I call $(\partial_\pm x)^\beta \equiv J(A)_\nu^\beta \partial_\pm X^\nu$ following (52)¹³.

In the reduced framework it is easy to convince ourselves that the only T-dual change for $(\partial_\pm x)$ that transforms the pull-back diagonally is nothing just the one responsible for the Buscher 's formulas :

$$\partial_\pm \tilde{X}^\mu = T_\pm^\mu{}_\nu \partial_\pm X^\nu \quad (54)$$

being the reduced pullback transformation

$$\begin{aligned} \tilde{\gamma}_{\pm 0}^0 &= -\gamma_{\pm 0}^0 \\ \tilde{\gamma}_{\pm i}^0 &= \mp G_{00} \gamma_{\pm i}^0 \\ \tilde{\gamma}_{\pm 0}^i &= \mp \frac{1}{G_{00}} \gamma_{\pm 0}^i \\ \tilde{\gamma}_{\pm j}^i &= \frac{1}{G_{00}} \gamma_{\pm j}^i \end{aligned} \quad (55)$$

which again can be summarized

$$\tilde{\gamma}_{\pm\mu}^\nu = (\mp G_{00})^{(n^0-n_0)} \gamma_{\pm\mu}^\nu \quad (56)$$

except for the $\gamma_{\pm 0}^0$, in which there is a flip of sign allowing the T-duality covariantization (35) of the world-sheet covariant derivatives.

It is worthwhile to mention (54) implies $K_\mu \nabla_+ \nabla_- X^\mu = K_\mu \nabla_- \nabla_+ X^\mu = 0 \longleftrightarrow \tilde{K}_\mu \tilde{\nabla}_+ \tilde{\nabla}_- \tilde{X}^\mu = \tilde{K}_\mu \tilde{\nabla}_- \tilde{\nabla}_+ \tilde{X}^\mu = 0$, which is automatically valid for the classical string (it is the current conservation corresponding to the isometry).

Again the simplicity of (55) allow us to build the diagonal mapping for any $Y(\sigma^+, \sigma^-)$ mapping¹⁴ :

$$\tilde{y}^\mp(\sigma^+, \sigma^-)_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l} = (\mp G_{00})^{(n^0-n_0)} y(\sigma^+, \sigma^-)_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l} \quad (57)$$

with the property of transforming linearly the covariant derivatives ∇_\pm :

$$\tilde{\nabla}_\pm \tilde{y}(\sigma^+, \sigma^-)_{\nu_1, \dots, \nu_m}^{\mp\mu_1, \dots, \mu_l} = (\mp G_{00})^{(n^0-n_0)} \nabla_\pm y(\sigma^+, \sigma^-)_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_l} \quad (58)$$

¹³I write the reduced ∂X^μ between parentheses to specify that in general it is not the partial derivative of anything. Only when the pull-back of $F(A)_{ij}$ vanishes x^μ can be identified with a sort of U(1) invariant embedding coordinates.

¹⁴The intermediate tool is now the analogue to (35) in the world-sheet, i.e., $d_\pm \Theta(\sigma^+, \sigma^-) \equiv (\partial_\pm + \Delta_\Theta \gamma_{\pm 0}^0) \Theta(\sigma^+, \sigma^-)$, transforming as Θ with weight Δ_Θ under T-duality.

In the usual setting (57) and (58) can be written with the help of the preceding work (38) (39) and (44) (45) as

$$\begin{aligned}\tilde{Y}(\sigma^+, \sigma^-)^{\pm\mu_1, \dots, \mu_l}_{\nu_1, \dots, \nu_m} &= \left(\prod_{r=1}^l T_{\pm\beta_r}^{\mu_r}\right) \left(\prod_{s=1}^m T_{\nu_s}^{\pm\alpha_s}\right) Y(\sigma^+, \sigma^-)^{\beta_1, \dots, \beta_l}_{\alpha_1, \dots, \alpha_m} \\ \tilde{\nabla}_{\mp} \tilde{Y}^{\pm\mu_1, \dots, \mu_l}_{\nu_1, \dots, \nu_m} &= \left(\prod_{r=1}^l T_{\pm\beta_r}^{\mu_r}\right) \left(\prod_{s=1}^m T_{\nu_s}^{\pm\alpha_s}\right) \nabla_{\mp} Y^{\beta_1, \dots, \beta_l}_{\alpha_1, \dots, \alpha_m}\end{aligned}\quad (59)$$

(59) allow us to extract the pull-back's T-duality transformation in the common language:

$$\tilde{\Gamma}_{\pm\mu}^{\rho} = T_{\mu}^{\mp\lambda} T_{\mp\beta}^{\rho} \Gamma_{\pm\lambda}^{\beta} + (\partial_{\pm} T_{\mu}^{\mp\beta}) T_{\mp\beta}^{\rho} \quad (60)$$

The most relevant example of this mapping is provided by the $\partial_{\pm} X^{\mu}$ for which holds $\tilde{\nabla}_{\pm} \partial_{\mp} \tilde{X}^{\mu} = T_{\mp\nu}^{\mu} \nabla_{\pm} \partial_{\mp} X^{\nu}$, giving the classical stringy equivalence between the two different geometries.

5 Generalized Curvature's Transformation

Defining as usual the generalized curvature as $R_{\mu\nu\sigma\rho}^{\pm} = R(\Gamma^{\pm})_{\mu\nu\sigma}^{\lambda} G_{\lambda\rho}$ which has the symmetry properties $R_{\mu\nu\sigma\rho}^{\pm} = -R_{\nu\mu\sigma\rho}^{\pm} = -R_{\mu\nu\rho\sigma}^{\pm}$, $R_{\mu\nu\rho\sigma}^{\pm} = R_{\rho\sigma\mu\nu}^{\mp}$ and taking account the generalized connection's transformation (33) ¹⁵ we get the dual generalized curvature in the reduced framework :

$$\begin{aligned}\tilde{r}_{0i0j}^{\pm} &= -\frac{1}{(G_{00})^2} (r_{0i0j}^{\pm} - \frac{1}{2G_{00}} \partial_i G_{00} \partial_j G_{00}) \\ \tilde{r}_{0ijk}^{\pm} &= \mp \frac{1}{G_{00}} (r_{0ijk}^{\pm} - \partial_i G_{00} \gamma_{jk}^{\mp 0}) \\ \tilde{r}_{ijk0}^{\pm} &= \pm \frac{1}{G_{00}} (r_{ijk0}^{\pm} + \partial_k G_{00} \gamma_{ij}^{\pm 0}) \\ \tilde{r}_{ijkl}^{\pm} &= r_{ijkl}^{\pm} - 2G_{00} \gamma_{ij}^{\pm 0} \gamma_{kl}^{\mp 0}\end{aligned}\quad (61)$$

We can convince ourselves the inhomogeneous part of the transformation can be written in terms of the Killing vector, giving the compact result :

$$\tilde{r}_{\mu\nu\sigma\rho}^{\pm} = (-1)^{(g_{\mu}+g_{\nu})} (\pm G_{00})^{-n_0} (r_{\mu\nu\sigma\rho}^{\pm} - \frac{2}{k^2} \nabla_{\mu}^{\pm} k_{\nu} \nabla_{\sigma}^{\mp} k_{\rho}) \quad (62)$$

where $K^2 = k^2 = K_{\mu} K^{\mu} = G_{00}$. In the usual setting the transformation (62) reads ¹⁶

$$\tilde{R}_{\mu\nu\sigma\rho}^{\pm} = T_{\mu}^{\mp\alpha} T_{\nu}^{\mp\beta} T_{\sigma}^{\pm\delta} T_{\rho}^{\pm\eta} (R_{\alpha\beta\delta\eta}^{\pm} - \frac{2}{K^2} \nabla_{\alpha}^{\pm} K_{\beta} \nabla_{\delta}^{\mp} K_{\eta}) \quad (63)$$

¹⁵See Apendix for explicit formulas

¹⁶ $\nabla_{\mu}^{\pm} k_{\nu}$ only has antisymmetric part due to the Killing condition.

It is important to note that the inhomogeneous part of the transformation only depends on the transverse components $G_{0\mu}, B_{0i}$ and their first skew-symmetric derivatives.

When an object transforms inhomogeneously, say $\tilde{r} = \pm(G_{00})^\Delta(r + \psi)$, the involution property of the T-duality transformation $T^2 = 1$ fixes the ψ transformation to be $\tilde{\psi} = \mp(G_{00})^\Delta\psi$, allowing to create the homogeneous $w \equiv r + \frac{1}{2}\psi$ i.e., $\tilde{w} = \pm(G_{00})^\Delta w$. Specifically, it means

$$\frac{1}{\tilde{K}^2} \tilde{\nabla}_\mu^\pm \tilde{K}_\nu \tilde{\nabla}_\sigma^\mp \tilde{K}_\rho = -T_\mu^\mp{}^\alpha T_\nu^\mp{}^\beta T_\sigma^\pm{}^\delta T_\rho^\pm{}^\eta \left(\frac{1}{K^2} \nabla_\alpha^\pm K_\beta \nabla_\delta^\mp K_\eta \right) \quad (64)$$

and therefore we can create the "corrected" generalized curvature $W_{\mu\nu\sigma\rho}^\pm$ transforming linearly under T-duality :

$$\begin{aligned} W_{\mu\nu\sigma\rho}^\pm &\equiv R_{\mu\nu\sigma\rho}^\pm - \frac{1}{K^2} \nabla_\mu^\pm K_\nu \nabla_\sigma^\mp K_\rho \\ \tilde{W}_{\mu\nu\sigma\rho}^\pm &= T_\mu^\mp{}^\alpha T_\nu^\mp{}^\beta T_\sigma^\pm{}^\delta T_\rho^\pm{}^\eta W_{\alpha\beta\delta\eta}^\pm \end{aligned} \quad (65)$$

At this point is for free the rederivation of the generalized Ricci-tensor's transformation giving the need of a non-trivial dilaton change under T-duality in order to guarantee the one-loop conformal invariance of the dual string sigma-model [2].

We get the dual-reduced-Ricci-tensor from (61) $r_{\mu\nu}^\pm$:

$$\tilde{r}_{\mu\nu}^\pm = (-1)^{g_\mu} (\pm G_{00})^{-n_0} (r_{\mu\nu}^\pm - \nabla_\mu^\pm \nabla_\nu^\pm \ln G_{00}) \quad (66)$$

It is well known the sigma-model coupling between the two-dimensional curvature and an scalar field called dilaton, Φ , it is enough to ensure the vanishing of the dual one-loop beta function, provided the former transforms under T-duality as

$$\tilde{\Phi} = \Phi - \frac{1}{2} \ln G_{00} \quad (67)$$

In other words, the tensor representing the one-loop beta function for the string sigma-model (bosonic and supersymmetric) [11], ie, $\beta_{\mu\nu}^\pm = R_{\mu\nu}^\pm - 2\nabla_\mu^\pm \nabla_\nu^\pm \Phi$, transforms linearly under T-duality as can be read off from (66) and (67) :

$$\tilde{\beta}_{\mu\nu}^\pm = T_\mu^\mp{}^\alpha T_\nu^\mp{}^\lambda \beta_{\alpha\lambda}^\pm \quad (68)$$

Therefore the vanishing of $\beta_{\mu\nu}$ implies the one for $\tilde{\beta}_{\mu\nu}$ and vice versa. Following this approach, the dilaton one-loop beta function can be obtained trying to complete minimally the generalized scalar curvature R^\pm in order to get a T-duality scalar ; the transformation for $R^\pm = r^\pm$ results to be (61) (66) :

$$\tilde{R}^\pm = \tilde{r}^\pm = R^\pm - 2\hat{\nabla}^{\pm i} \partial_i \ln G_{00} + 2 \frac{1}{G_{00}} \gamma_{i0}^{\pm k} \gamma_{0k}^{\pm i} \quad (69)$$

and the desired T-duality scalar is

$$\beta^\Phi = R^\pm + 4((\partial\Phi)^2 - (\nabla^\pm)^2\Phi) - \frac{2}{3}H^2 \quad (70)$$

where we define $H_{\alpha\beta}^2 \equiv H_{\alpha\nu\rho} H_\beta^{\nu\rho}$. Modulo a constant term it coincides with the one-loop dilaton beta-function [11].

6 Canonical Connection and Curvature

The anomalous transformation of the index corresponding with first covariant derivations propagates in an annoying way to objects constructed from higher derivations, such as the generalized curvatures and the one-loop beta functions. In other words, the covariant derivation ∇^\pm does not commute with the canonical map defined in (44).

Looking at the generalized connections' transformation (33) we notice there is a minimal covariant subtraction giving a connection, the canonical connection, which commutes with the canonical T-duality:

$$\begin{aligned}\bar{\Gamma}_{\mu\nu}^{\pm\rho} &\equiv \Gamma_{\mu\nu}^{\pm\rho} - \frac{1}{K^2} K_\mu \nabla_\nu^\mp K^\rho \\ \tilde{\Gamma}_{\mu\nu}^{\pm\rho} &= T_\mu^{\pm\lambda} T_\nu^{\pm\beta} T_{\pm\alpha}^\rho \bar{\Gamma}_{\lambda\beta}^{\pm\alpha} + (\partial_\mu T_\nu^{\pm\beta}) T_{\pm\beta}^\rho\end{aligned}\quad (71)$$

Because the anomaly mentioned above is located in the $\gamma_{0\nu}^{\pm\sigma}$ components, the whole effect of the subtraction is to cancel against them, giving $\bar{\gamma}_{0\mu}^{\pm\rho} = 0$ and $\bar{\gamma}_{i\nu}^{\pm\rho} = \gamma_{i\nu}^{\pm\rho}$ ¹⁷. This fact allows the commutation between $\bar{\nabla}_\mu^\pm$ and the canonical T-duality map :

$$\begin{aligned}\tilde{V}_{\nu_1, \dots, \nu_m}^{\pm\mu_1, \dots, \mu_l} &= \left(\prod_{r=1}^l T_{\pm\beta_r}^{\mu_r} \prod_{s=1}^m T_{\nu_s}^{\pm\alpha_s} \right) V_{\alpha_1, \dots, \alpha_m}^{\beta_1, \dots, \beta_l} \\ \bar{\nabla}_\rho^\pm \tilde{V}_{\nu_1, \dots, \nu_m}^{\pm\mu_1, \dots, \mu_l} &= T_\rho^{\pm\lambda} \left(\prod_{r=1}^l T_{\pm\beta_r}^{\mu_r} \right) \left(\prod_{s=1}^m T_{\nu_s}^{\pm\alpha_s} \right) \bar{\nabla}_\lambda^\pm V_{\alpha_1, \dots, \alpha_m}^{\beta_1, \dots, \beta_l}\end{aligned}\quad (72)$$

Another consequence is $\bar{\nabla}_0^\pm = 0$ implying $\bar{R}_{0\nu\sigma\rho}^\pm = 0$. Even more, the new connection is compatible with the metric provided that K^μ is a Killing vector.

In a certain sense the barred connection seems to be the most natural associated to the presence of a Killing. If we think the Killing as a vector field indicating the direction in which nothing changes, we would expect the parallel transport is really insensitive to displacements in that direction. This happens with the canonical connection but not with the Levi-Civita (or its torsionfull generalizations) one.

The commutation with the canonical T-duality implies that the curvature for $\bar{\nabla}^\pm$ transforms canonically (the same as the Ricci tensor and the scalar curvature).

$$\tilde{\bar{R}}_{\mu\nu\sigma\rho}^\pm = T_\mu^{\pm\alpha} T_\nu^{\pm\beta} T_\sigma^{\pm\delta} T_\rho^{\pm\eta} \bar{R}_{\alpha\beta\delta\eta}^\pm \quad (73)$$

I will list five independent T-duality scalars

$$\begin{aligned}I_1 &= R^\pm - (\nabla^\pm)^2 \ln K^2 - \frac{2}{K^2} H_{\alpha\beta}^2 K^\alpha K^\beta \\ I_2 &= H^2 - \frac{3}{K^2} H_{\alpha\beta}^2 K^\alpha K^\beta \\ I_3 &= \frac{1}{K^2} (\nabla_\mu K_\nu \nabla^\mu K^\nu + H_{\alpha\beta}^2 K^\alpha K^\beta)\end{aligned}$$

¹⁷ These conditions guarantee that if a tensor is covariantly constant with respect to ∇_μ^\pm it does too with respect to $\bar{\nabla}_\mu^\pm$. The converse is in general not true. Therefore the metric commutes with $\bar{\nabla}_\mu^\pm$

$$\begin{aligned}
I_4 &= \frac{1}{K^2} K_\alpha \partial_\beta K_\lambda H^{\alpha\beta\lambda} \\
I_5 &= \left(\frac{\partial K^2}{K^2} \right)^2
\end{aligned} \tag{74}$$

built with the help of \bar{R}^\pm and being ∇_μ the Levi-Civita covariant derivation. In particular, $\bar{R}^\pm = I_1 \mp 2I_4$.

Finally, with this covariant derivation, the T-duality canonical map commutes with the basic geometrical operations: linear combinations, tensor products, permutation of indices, contractions and covariant derivations.

7 Conclusions

This work shows how the reduced geometry is a privileged framework to the study of the T-duality's geometry and possibly of many other different issues related with the abelian Killing vectors.

The T-duality transformation diagonalizes in the reduced setting, allowing us to get in a straightforward way results pursued since a long time, such as the generalized curvatures' map, the canonical map for the covariant derivatives in the target-space and in the world-sheet, the minimal correction to connections and curvatures in order to transform linearly and T-duality scalars. The introduction of the dilaton can be seen as the minimal modification needed to map the 1-loop beta-functions preserving conformal invariance, but from the geometrical view, the dilaton is completely insufficient to build in a systematic way T-duality tensors (i.e., tensors transforming canonically under T-duality). The object serving to "covariantize" under T-duality is the Killing vector itself across the canonical connection, as it was shown in the last section. In connection to that, new T-duality scalars have been found without the help of the dilaton.

Future work could be the higher-loop corrections to the Buscher's formulas, the map for the invariants characterizing the geometry and the topology of the manifold, the global questions in the reduced setting, the non-abelian generalizations of this procedure, and the deeper study of the geometry of the canonical connection and its relation with the string sigma model.

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A Apendix : The reduced connection and Riemaniann curvature

In order to have the opportunity of getting an idea about the operativity of the reduced framework, I show here the usual Levi-Civita connection for the generic metric with n-commuting Killing vector fields :

$$\begin{aligned}
G_{\mu\nu} &= \begin{pmatrix} G_{ab} & A_{ai} \\ A_{bj} & \hat{G}_{ij} + A_{ic}A_{jd}G^{cd} \end{pmatrix} \\
\Gamma_{bc}^a &= \frac{1}{2}A_i^a\hat{\partial}^iG_{bc} \\
\Gamma_{ab}^i &= -\frac{1}{2}\hat{\partial}^iG_{ab} \\
\Gamma_{ai}^j &= \Gamma_{ia}^j = \frac{1}{2}(G_{ab}\hat{F}_i^{bj} - A_i^b\hat{\partial}^jG_{ab}) \\
\Gamma_{ai}^b &= \Gamma_{ia}^b = \frac{1}{2}(G^{bc}\partial_iG_{ac} - A_j^bG_{ac}\hat{F}_i^{cj} + A_i^cA_j^b\hat{\partial}^jG_{ac}) \\
\Gamma_{ij}^k &= \hat{\Gamma}_{ij}^k + \frac{1}{2}G_{ab}(A_j^b\hat{F}_i^{ak} + A_i^b\hat{F}_j^{ak}) - \frac{1}{2}A_i^aA_j^b\hat{\partial}^kG_{ab} \\
\Gamma_{ij}^a &= \frac{1}{2}\{(\hat{\nabla}_iA_j^a + \hat{\nabla}_jA_i^a) + A_i^cA_j^bA_k^a\hat{\partial}^kG_{bc} + G^{ab}(A_j^c\partial_iG_{bc} + A_i^c\partial_jG_{bc}) - A_k^aG_{cb}(A_j^b\hat{F}_i^{ck} + A_i^b\hat{F}_j^{ck})\}
\end{aligned} \tag{75}$$

The (75) expressions must be compared with (18) to realize the advantages.

Now I would like to make explicit the relation between a generic connection Γ and its reduced version γ :

$$\begin{aligned}
\gamma_{ab}^i &= \Gamma_{ab}^i \\
\gamma_{ab}^c &= \Gamma_{ab}^c + \Gamma_{ab}^iA_i^c \\
\gamma_{ai}^j &= \Gamma_{ai}^j - \Gamma_{ab}^jA_i^b \\
\gamma_{ia}^j &= \Gamma_{ia}^j - \Gamma_{ba}^jA_i^b \\
\gamma_{ia}^b &= \Gamma_{ia}^b - \Gamma_{ca}^bA_i^c + \Gamma_{ia}^jA_j^b - \Gamma_{ca}^jA_i^cA_j^b \\
\gamma_{ai}^b &= \Gamma_{ai}^b - \Gamma_{ac}^bA_i^c + \Gamma_{ai}^jA_j^b - \Gamma_{ac}^jA_i^cA_j^b \\
\gamma_{ij}^k &= \Gamma_{ij}^k - \Gamma_{aj}^kA_i^a - \Gamma_{ia}^kA_j^a + \Gamma_{ab}^kA_i^aA_j^b \\
\gamma_{ij}^a &= \Gamma_{ij}^a - \Gamma_{bj}^aA_i^b - \Gamma_{ib}^aA_j^b + \Gamma_{bc}^aA_i^bA_j^c - \Gamma_{bj}^kA_i^bA_k^a - \Gamma_{ib}^kA_j^bA_k^a + \Gamma_{bc}^kA_i^bA_j^cA_k^a \\
&\quad - \partial_iA_j^a + \Gamma_{ij}^kA_k^a
\end{aligned} \tag{76}$$

Paradoxical though it may be seen the task to calculate the reduced Levi-Civita connection is shorter than the usual one, because the knowledge of its invariance under the adapted diffeomorphisms (6) drops out the terms proportionally to A_i^a being non derivatives (we can call it $A = 0$ projection). Therefore (76) must be understood as $\gamma_{\mu\nu}^\rho = (\Gamma_{\mu\nu}^\rho - \partial_\mu J_\nu^\rho)|_{A=0}$. It is just the need to know $\Gamma|_{A=0}$ instead of Γ itself, which makes easier to get the reduced version.

The usual curvature, torsion and Ricci tensor's expression for a generic connection $\Gamma_{\mu\nu}^\rho$ are :

$$\begin{aligned}
R(\Gamma)_{\mu\nu\sigma}^\rho &= \partial_\mu\Gamma_{\nu\sigma}^\rho - \partial_\nu\Gamma_{\mu\sigma}^\rho - \Gamma_{\mu\sigma}^\beta\Gamma_{\nu\beta}^\rho + \Gamma_{\nu\sigma}^\beta\Gamma_{\mu\beta}^\rho \\
T(\Gamma)_{\mu\nu}^\rho &= \frac{1}{2}(\Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho) \\
R(\Gamma)_{\mu\nu} &= R(\Gamma)_{\mu\rho\nu}^\rho
\end{aligned} \tag{77}$$

The reduced Riemaniann curvature for the $U(1)^n$ is given by

$$\begin{aligned}
r_{abcd} &= \frac{1}{4}(\hat{\partial}^i G_{ac} \partial_i G_{bd} - \hat{\partial}^i G_{bc} \partial_i G_{ad}) \\
r_{abci} &= \frac{1}{4} F_{ij}^e \hat{\partial}^j G_{cd} (G_{ea} \delta_b^d - G_{eb} \delta_a^d) \\
r_{ijab} &= \frac{1}{4} (G^{cd} \partial_i G_{ac} \partial_j G_{bd} + G_{ac} G_{bd} \hat{F}_i^{ck} F_{jk}^d - (i \leftrightarrow j)) \\
r_{aibj} &= \frac{1}{2} \hat{\nabla}_i \hat{\nabla}_j G_{ab} - \frac{1}{4} G^{cd} \partial_i G_{bd} \partial_j G_{ac} - \frac{1}{4} G_{ac} G_{bd} \hat{F}_i^{dk} F_{jk}^c \\
r_{akij} &= -\frac{1}{2} \hat{\nabla}_k (G_{ab} F_{ij}^b) + \frac{1}{4} (F_{ki}^b \partial_j G_{ab} - F_{kj}^b \partial_i G_{ab}) \\
r_{ijkl} &= \hat{R}_{ilkl} + \frac{1}{4} G_{ab} (F_{ik}^a F_{jl}^b - F_{jk}^a F_{il}^b + 2 F_{ij}^a F_{kl}^b)
\end{aligned} \tag{78}$$

where the hatted objects are the ones calculated with the quotient metric \bar{G}_{ij} , $F_{ij}^a \equiv \partial_i A_j^a - \partial_j A_i^a$, and $A_i^a = G^{ab} A_{ib}$. The Riemaniann curvature is obtained using (12)(8)(9), i.e. $R = J(-A)r$.

B Appendix :The generalized curvature

In this section I will write the reduced generalized curvature needed for the fifth section's calculations. To do it in a T-duality suitable way means the introduction of the $\hat{D}_i^\pm \Theta \equiv \hat{\nabla}_i^\pm + \frac{\Delta_\Theta}{2} \partial_i \ln G_{00} \Theta$ T-duality covariant derivative. As in (35) Δ_Θ is the Θ 's T-duality weight.

In terms of G_{00} and $F_{ij}^\pm \equiv \gamma_{ij}^{\pm 0}$ the relevant components are:

$$\begin{aligned}
r_{0i0j}^\pm &= \left\{ \frac{1}{2} \hat{D}_i^\pm \partial_j G_{00} + G_{00}^2 \hat{F}_i^{\pm k} F_{kj}^\mp \right\} + \frac{G_{00}}{4} \partial_i \ln G_{00} \partial_j \ln G_{00} \\
r_{0ijk}^\pm &= \left\{ G_{00} \hat{D}_i^\pm F_{jk}^\mp + \frac{G_{00}}{2} (F_{ik}^\pm \partial_j \ln G_{00} - F_{ij}^\pm \partial_k \ln G_{00}) \right\} + \frac{G_{00}}{2} F_{jk}^\mp \partial_i \ln G_{00} \\
r_{ijkl}^\pm &= \left\{ \hat{R}_{ijkl}^\pm + G_{00} (F_{ik}^\pm F_{jl}^\pm - F_{jk}^\pm F_{il}^\pm) \right\} + G_{00} (\gamma_{ij}^\pm + \gamma_{ij}^\mp) \gamma_{kl}^\mp
\end{aligned} \tag{79}$$

where $\hat{R}_{ijkl}^\pm = R(\hat{\Gamma} \pm \hat{h})_{ijkl}$. The other components are related by the symmetry properties $R_{\mu\nu\sigma\rho}^\pm = -R_{\nu\mu\sigma\rho}^\pm = -R_{\mu\nu\rho\sigma}^\pm$ and $R_{\mu\nu\rho\sigma}^\pm = R_{\rho\sigma\mu\nu}^\mp$. The terms in brackets are the ones transforming with the dominant $(-1)^{(g_\mu + g_\nu)} (\pm G_{00})^{-n_0}$, while the remaining terms transform with the $(-1)^{(g_\mu + g_\nu)} (\pm G_{00})^{-n_0}$ giving the inhomogeneous part $-\frac{2}{K^2} \nabla_\mu^\pm K_\nu \nabla_\sigma^\mp K_\rho$.

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